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SEPARATION OF A FINITE SET IN $\mathbb{R}^d$ BY SPANNED HYPERPLANES*

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A question of the following kind will concern us here: what is the minimal number n, ensuring that any spanning set of n points in 3-space spans a plane, every open side of which contains at least, say, 1000 points of the set. The answer is $n=4001$ (see Theorem 2.1 below).

1. Introduction

Let V be a finite subset of an arbitrary euclidean space. Denote the affine flat spanned by V by $\text{aff}(V)$. A flat J is a *spanned hyperplane of* V if (i) $J=\text{aff}(V\cap J)$, and (ii) $\dim J=\dim(\text{aff}(V))-1$. Assume that dim $\text{aff}(V)=d$. If $d\geq 1, J$ separates $\text{aff}(V)$ into two open half-spaces. If one of them contains at least k points of V and the other contains at least ℓ points of V, we say that J is of *type* $(k,\ell)(k,\ell$ are non-negative integers). In case J is of type (k,k) (i.e., $k=\ell$) it is a *k-bisector.* J is of *type* $(s)(s>1)$ if $\#(V\setminus J)\geq s$. Hence being of type (s) implies being of type (k,ℓ) for some $k,\ell\geq 0$ with $k+\ell\geq s$. Let T denote one of the types (k,ℓ) or (s) defined above. V is of *type* T if there is a hyperplane of type T spanned by V. Define

$$(1.1) \qquad f(T;d)=\sup\{\#V : V\subset\mathbb{R}^d, \text{aff}(V)=\mathbb{R}^d, \qquad V \quad \text{is } \textit{not} \text{ of type} \quad T\}.$$

The "sup" in this definition has to be interpreted as follows: If there are arbitrary large finite spanning sets in $\mathbb{R}^d$, which are not of type T, then $f(T;d)=\infty$. Otherwise, if there are spanning subsets which are not of type T, then $f(T;d)$ is the maximal size. If *every* spanning subset in $\mathbb{R}^d$ is of type T, then we use the convention $f(T;d)=d$; e.g., $f(0,1;2)=f(1;2)=2$ by this convention. That $f(T;d)$ is finite for all k,ℓ,s,d can be shown easily, e.g., by induction on d. A set V in $\mathbb{R}^d$ which is not of type T, of maximal size (i.e., of size $\#V=f(T;d)$) is called an *extremal set.*

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The functions $f(s;d)$ and $f((0,k);d)$ are discussed in [3] and [4], respectively. Here we are mainly interested in $f(k,k;d)$. In particular we show that $f(k,k;3)=4k$ for $k\geq 1$ (Theorem 2.1) and thereby describe corresponding extremal sets (Proposition 2.5 below).

2. A proof of $f(k,k;3)=4k$ and related results

Theorem 2.1. $f(k,k;3)=4k$.

Proof. To prove "$\geq$" take two skew lines in $\mathbb{R}^3$, and choose on each one of them $2k$ points. The resulting set of $4k$ points has the property that any plane spanned by it is even not of type $(2k)$, less it can be of type (k,k). (Proposition 2.5 below says that, essentially, this is the only extremal set).

Proof of "$\leq$" : Let $V\subset\mathbb{R}^3$ be a spanning set, $\#V>4k$, and assume that V is not of type (k,k) (we have to contradict this assumption). A closed half space is *strong* if its boundary plane is spanned by V, and its complement (an open half space) contains less then k points of V (note that both closed sides of a spanned plane can be strong). Clearly the intersection of every 4 strong half-spaces contains at least

$$4k+1-4(k-1)=5 \tag{2.1}$$

points of V. (Note that there are at least 4 strong half-spaces since every facet of the convex hull [V] of V determines one.) Clearly every spanned plane which is not of type (k,k) determines at least one strong half-space. Let Φ be a family of strong half spaces with the following properties:

(i) Φ contains a *unique* strong half space versus every spanned plane which is not of type (k,k), and

(ii) Φ is maximal (under inclusion).

By (2.1) and Helly's Theorem $\cap\Phi\neq\emptyset$, hence $0\leq\dim\cap\Phi\leq 3$ (clearly $\cap\Phi\subset[V]$). If $\dim\cap\Phi=3$, choose $z\in\cap\Phi$ s.t. z does not lie on any plane spanned by V. By Charathéodory Lemma z is an inner point of a tetrahedron whose vertices are in V. Let Δ be such a tetrahedron which is minimal (under inclusion); we get a contradiction to (2.1), because, by assumption, every facet of Δ is not of type (k,k), and it determines a closed half space belonging to Φ (Φ is maximal), hence this half-space contains z, so the intersection of the 4 strong half spaces in Φ determined by the facets of Δ is precisely Δ, hence it contains only 4 points of V (the vertices of Δ). This contradicts (2.1).

Corollary. $0\leq\dim\cap\Phi<3$.

Φ has a minimal subfamily Θ s.t. $\dim\cap\Theta<3$. Put $t=\#\Theta$; clearly $t\geq 2$ (otherwise $\dim\cap\Theta=3$), and in fact $t\geq 3$ (if $t=2$, then $\dim\cap\Theta<3$ implies $\dim\cap\Theta=2$, thus Θ consists of two *opposite* strong half spaces, contrary to the requirement (i) for Φ).

Since $\dim\cap\Theta\geq 0$, it follows easily that

$$t=\#\Theta=3+1-\dim\cap\Theta\leq 3+1-0=4, \tag{2.2}$$

i.e., $3 \le t \le 4$. Denote by $G_i^+ \quad 1 \le i \le t$ the members of Θ. It is easily seen that $\bigcap_{i=1}^t G_i^+ = \bigcap_{i=1}^t \mathrm{bd} G_i^+$ and this set is a flat of dimension $3+1-t=4-t$.

If $t=4$, then $\bigcap_{i=1}^4 G_i^+$ is 0-dimensional, i.e., it is a one point intersection of 4 strong half spaces, contradiction to (2.1).

Hence $\#\Theta = t = 3$ and $\dim \cap \Theta = 4-t = 4-3 = 1$, i.e., $\cap\Theta$ is a line; denote this line by L. Put $G_i^- = \mathbb{R}^3 \setminus G_i^+$; then $\#(V \cap G_i^-) \le k-1$, hence

$$\begin{aligned} \#(V \cap L) &= \#(V \cap (\cap\Theta)) = \#(V \setminus (\cup_{i=1}^3 G_i^-)) \\ &\ge \#V - \sum_{i=1}^{3} \#(V \cap G_i^-) \ge 4k+1-3(k-1) = k+4. \end{aligned} \tag{2.3}$$

Our aim now is to find a tetrahedron Δ s.t. $V \cap \Delta = \mathrm{vert}\Delta$ (hence $\#(V \cap \Delta) = 4$) and the closed half spaces determined by the facets of Δ containing Δ are strong; this will contradict (2.1).

Put $n = \#(V \cap L)(n \ge k+4)$ and let $p_1, \cdots, p_n$ be the points of $V \cap L$, indexed according to their linear order on L. Let $\hat{\Phi}$ be the set of all strong half-spaces spanned by $V(\Phi \subset \hat{\Phi}, \cap\hat{\Phi} \ne \emptyset$ by (2.1)).

There is a segment $[p_{i-1}, p_{i+1}] \quad 2 \le i \le n-1$ whose relative interior contains a point of $\cap\hat{\Phi}$ (actually, there is such a segment of type $[p_i, p_{i+1}] \quad 1 \le i \le n-1$, unless $\cap\hat{\Phi}$ consists of one point only, which is in V).

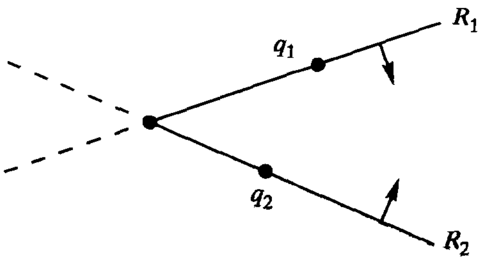

Fig. 1

Consider all the planes spanned by L and one more point of V; project these planes on a plane perpendicular to L. We get a figure resembling a planar Gale Diagram. Draw boldly the projections of the half planes bounded by L containing one more point of V (see fig. 1). This yields a set of "bold" rays (probably some couples of them are colinear and opposite to each other).

To every bold ray draw a small perpendicular arrow, pointing to any strong half space (in $\hat{\Phi}$) whose boundary is the plane the projection of which contains that ray. Note that a-priori there can be two (opposite) arrows attached to one bold ray.

If there are two successive bold ray R_1, R_2 (i.e., not opposite to each other and there is no other bold ray between them), whose arrows point to each other (like in fig. 1), then construct Δ as follows: choose a point $q_j (j=1,2)$ of V on the half plane whose projection is the bold ray R_j, which is closest to L, and choose the points p_{i-1}, p_{i+1} of L. All the half spaces determined by the facets of the tetrahedron $[p_{i-1}, p_{i+1}, q_1, q_2]$ which contain it are strong; the plane aff$\{q_1, q_2, p_i\}$ divides this tetrahedron into two tetrahedrons, one of which is appropriate to be Δ: only its vertices are in V, and all the half spaces determined by its facets and contain it are strong.

It remains to show that there are two successive bold rays whose arrows point to each other. It suffices to show that the arrows are not all one sidely oriented, say counterclockwise. If there are two bold rays which are opposite to each other, then they have arrows pointing to the same side of the line determined by them, and in this side there must be another bold ray, and the remaining of the argument is simple (resembles the one dimensional case of Sperner's Lemma). So assume, w.l.o.g., that there are no two opposite bold rays. Similarly, we may assume that there is no bold ray with two (opposite) arrows. Divide all the rays (bold or thin) into groupings of bold and thin rays alternately. Since there is no couple of opposite rays of the same kind, the number of groupings of each kind (bold or thin)is odd, say $2r+1$. Assume now that all the arrows are one sidely oriented, say counterclockwise (fig. 2). (We have to refute this assumption.)

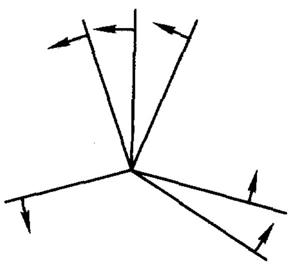

Fig. 2

Considering the last ray in any bold grouping shows that the union of the r bold groupings behind it contains *less* then k points of V, and considering the first ray of any bold grouping shows that the union of the r bold groupings in front of it contains *at least* k points of V (since there is no bold ray with two (opposite) arrows, by assumption). If this happens in every bold grouping, then every union of r successive bold groupings contains *less than* k points, and also *at least* k points; this yields $2r+1$ contradictions (!). ∎

Analysis of the proof 2.2. Let $V \subset \mathbb{R}^d (d \geq 1)$ be a spanning set. A d-simplex Δ with $V \cap \Delta = \text{vert}\Delta$ is a *cellular simplex* (relatively to V). The foregoing shows that if $V \subset \mathbb{R}^3, \#V \geq 4k$, and if V is not of type (k,k) then there is a cellular simplex Δ, the facets of which determine strong half spaces containing Δ (and if $\#V > 4k$ this contradicts (2.1)). In relation to this, define, for all $d \geq 1$ and a spanning set $V \subset \mathbb{R}^d$ a *k-point* (for V) as follows: it is a point p, not lying on any flat spanned by V, s.t. every cellular simplex Δ containing p has a facet which spans a hyperplane of type (k,k). In [2] the case $d=2$ of the following conjecture is proved.

Conjecture 2.3. *If $V \subset \mathbb{R}^d$ is a spanning set, $\#V > (d+1)k$ then V has a k-point. In particular, $f(k,k;d) \leq (d+1)k$ for all $k, d \geq 1$ (see also Prop. 3.1 below.)*

(Case $d = 1$ is trivial.) The foregoing proof is insufficient in itself to prove this conjecture even for $d = 3$, although it shows that "generically" (i.e., V being in general position) the conjecture is true; simple Helly types considerations of the above spirit yield:

Proposition 2.4. *If V is as in 2.3, and furthermore V is in general position, then V has a k-point (in particular, V has a k-bisector).*

Actually, the aforenamed considerations which the reader is urged to convince himself show that Proposition 2.4 stays true if "V is in general position" is replaced by "there are no $((d+1)k+1) - d(k-1) = k+d+1$ points of V on a spanned $(d-2)$-flat". We note, in passing, the following related result of [1] Th. 1: given a finite mass distribution in $\mathbb{R}^d$ there is a point $p \in \mathbb{R}^d$ such that any closed half-space containing p contains a portion of at least $\frac{1}{d+1}$ of the total mass.

The proof of Theorem 2.1 also helps in determining the extremal sets:

Proposition 2.5. *Let $V \subset \mathbb{R}^3$ be a spanning set not of type (k,k), and $\#V = 4k$. Then:*

(i) *for $k > 3$ V consists of two sets, each of cardinality $2k$, lying on two skew lines in $\mathbb{R}^3$.*

(ii) *For $k = 2$, there are some more possibilities described below, and for $k = 3$ there is only one more possibility depicted in fig. 4 below.*

Remark We refer to the set described in Proposition 2.5 (i) as the *standard example.*

Proof. As mentioned in 2.2, the proof of 2.1 yields a cellular tetrahedron Δ, the strong sides of all of whose facets contain Δ; this time no contradiction arises, as $\#V = 4k$. Let v_1, v_2, v_3, v_4 be the vertices of Δ, and denote by $H_i (1 \leq i \leq 4)$ the plane spanned by the facet of Δ which doesn't contain v_i. Let H_i^+ be the *open* side of H_i which doesn't contain v_i, and put $V_i = V \cap H_i^+$. By $\#V = 4k$, $V = \left(\bigcup_{i=1}^4 V_i\right) \cup \{v_1, v_2, v_3, v_4\}$ and since $\#V_i \leq k-1$ for $1 \leq i \leq 4$ it follows that $V_i \cap V_j = \emptyset$ for $1 \leq i < j \leq 4$, and that $\#V_i = k-1$ for $1 \leq i \leq 4$. Denote by $\hat{\Delta}_i (1 \leq i \leq 4)$ the (closed) trihedral angle of Δ in v_i; it follows that $V_i \subset \hat{\Delta}_i \cap H_i^+$ for $1 \leq i \leq 4$. Put $\hat{\Delta}_{ij} = \hat{\Delta}_i \cup \hat{\Delta}_j$ for $1 \leq i < j \leq 4$ ($\hat{\Delta}_{ij}$ is contained in the closed dihedral angle of Δ at the edge $[v_i, v_j]$).

Claim 1. $(\mathrm{int}\hat{\Delta}_{ij}) \cap V = \emptyset$ *for* $1 \leq i < j \leq 4$.

Proof. Assume, say, $v \in (\mathrm{int}\hat{\Delta}_{12}) \cap V$. To contradict this, consider the plane $\mathrm{aff}(v, v_1, v_2)$; on one open side of it there are $\#(V_3 \cup \{v_4\}) = k$ points of V, and on the other open side of it there are also $\#(V_4 \cup \{v_3\}) = k$ points of V, contradiction (proving the claim).

It follows that V is contained on the six lines spanned by the vertices of Δ. More precisely

$$V \setminus \mathrm{vert}\Delta \subset \cup\{\mathrm{aff}(v_i, v_j) \setminus [v_i, v_j] : 1 \leq i < j \leq 4\}.$$

For all $1 \leq i \neq j \leq 4$ denote by R_{ij} the open ray on $\mathrm{aff}(v_i, v_j)$, apexed at v_j and not containing v_i. The foregoing discussion shows that $\forall 1 \leq i \leq 4$

$$V_i \subset \cup\{R_{ij} : j \neq i, 1 \leq j \leq 4\}. \tag{2.5}$$

If $R_{ij} \cap V \neq \emptyset$ denote by v_{ij} the point of this set nearest to v_j.

Claim 2. *If $R_{ij} \cap V \neq \emptyset$, then $\#(R_{ij} \cap V) \geq k-2$ and $\#(R_{ji} \cap V) \geq k-2$.*

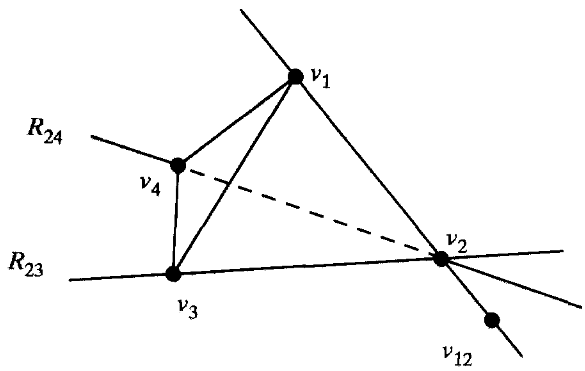

Fig. 3

Proof. (Fig. 3): For $k=1,2$ there is nothing to prove; so assume $k \geq 3$. Assume, w.l.o.g., that $i=1$ and $j=2$, i.e., $R_{12} \cap V \neq \emptyset$ (in particular, v_{12} is defined). Since $k \geq 3$ it suffices to prove that $\#(R_{21} \cap V) \geq k-2$ (this will show that, in particular, $R_{21} \cap V \neq \emptyset$, hence, $\#(R_{12} \cap V) \geq k-2$ will follow since the roles of the labels 1, 2 are, of course, interchangeable). So assume $\#(R_{21} \cap V) < k-2$ (we have to contradict this assumption). Then by $\#V_2 = k-1$ and (2.5)

$$\#((R_{23} \cup R_{24}) \cap V_2) \geq 2. \tag{2.6}$$

Hence, either $R_{23} \cap V \neq \emptyset$ or $R_{24} \cap V \neq \emptyset$. Assume, w.l.o.g., that $R_{23} \cap V \neq \emptyset$; in particular, v_{23} is defined. Put $H = \mathrm{aff}(v_3, v_4, v_{12})$. If either $\#(R_{34} \cap V) \leq 1$ or $\#(R_{43} \cap V) \leq 1$ then H is of type (k,k) (in one open side of H there are $k-2$ points of V_1 in addition to 2 points of V_2 (see (2.6)), and in the other open side there are v_1, v_2 in addition to at least $k-2$ points either of V_3 (if $\#(R_{34} \cap V) \leq 1$) or of V_4 (if $\#(R_{43} \cap V) \leq 1$). So we may assume that

$$\#(R_{34} \cap V) \geq 2 \quad \text{and} \quad \#(R_{43} \cap V) \geq 2. \tag{2.7}$$

Put $H'' = \mathrm{aff}(v_{23}, v_4, v_{12})$. The planes H and H'' share the line $\mathrm{aff}(v_4, v_{12})$, and it is quite clear from fig. 3 that H'' intersects R_{13} (this can be proved rigorously as follows: The planes H'' and $H^* := \mathrm{aff}(v_1, v_2, v_3)$ share the line $\mathrm{aff}(v_{23}, v_{12})$; so it suffices to show that R_{13} intersects $\mathrm{aff}(v_{23}, v_{12})$. It follows easily by the standard axioms of order in the plane H^* that R_{13} intersects $[v_{23}, v_{12}]$ (apply Pasch's axiom to the triangle $[v_{23}, v_2, v_{12}]$ intersected by the line $\mathrm{aff}(v_1, v_3)$). Hence if we rotate H around $\mathrm{aff}(v_4, v_{12})$ so as it always cuts R_{23} and R_{13} there is a first meeting with $(V_2 \cup V_1) \setminus \{v_{12}\}$ (in at most two points of this set), and this happens not later than the arrival of H to H''. Denote by H' the plane of this first meeting. We claim that H' is of type (k,k). Proof: One open side of H' contains at least one point of V_2 (by (2.6)), two points of R_{34} (by (2.7)) and $\#(V_1 \setminus \{v_{12}, v_{13}\}) = k-3$ points of V_1; which makes k points in all. The other open side of H' contains $V_4 \cup \{v_1, v_2, v_3\}$;which is $k+2$ points in all. This contradiction, proves claim 2.

Claim 3. *If $k > 3$ then V_i $(1 \leq i \leq 4)$ is contained in precisely one of the rays R_{ij} $j \neq i$, $1 \leq j \leq 4$.*

Proof. By r.a.a., assume, w.l.o.g., that $V_1 \cap R_{12} \neq \emptyset$ and $V_1 \cap R_{13} \neq \emptyset$. Then by claim 2 $\#V_1 \geq 2(k-2) = 2k-4 > k-1$ (since $k > 3$), contradiction (proving the claim).

Assume now, w.l.o.g., that $R_{12} \cap V \neq \emptyset$, and that $k > 3$; then by claims 2, 3 $V_1 \subset R_{12}$ and $V_2 \subset R_{21}$, so we have already $2k$ points on the line aff(v_1, v_2). We claim that all the remaining $2k$ points are on the line aff(v_3, v_4). For this it suffices, by the foregoing argument, to prove that $R_{34} \cap V \neq \emptyset$. But if $R_{34} \cap V = \emptyset$ then either $R_{31} \cap V \neq \emptyset$ or $R_{32} \cap V \neq \emptyset$. In the first case it follows from claims 2, 3 that $V_1 \subset R_{13}$ (contradiction), and in the second case we get $V_2 \subset R_{23}$ (contradiction.) This proves Proposition 2.5(i).

Regarding part (ii), note that (2.5) and $\#V_i = k - 1 (1 \leq i \leq 4)$ are valid for $k = 2, 3$ as well, and this leaves not too many possibilities. Without going into details, it turns out that for $k = 2$ (2.5) is not only necessary, but also sufficient for V not being of type (k, k); thus V is extremal for type (2,2) $(d = 3)$ if and only if there are 4 spanning points $v_1, v_2, v_3, v_4 \in V$ s.t. $\forall 1 \leq i \leq 4$ exactly *one* of the ray R_{ij} $1 \leq j \leq 4$ contains a point of V (the standard example is obviously included here). Combinatorially there are 9 or 10 types of this kind (which we didn't bother to list here).

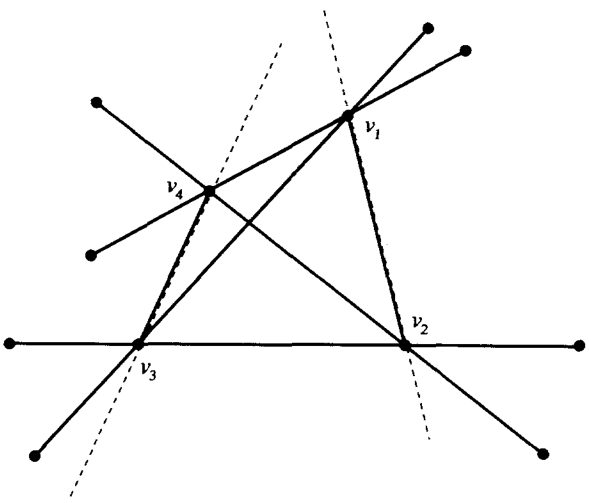

Fig. 4. The unique non-standard extremal set for $k = 3$

For $k = 3$ the situation is more restrictive. Here we can use claim 2, and it turns out, by argument of the foregoing kind, that for $k = 3$ besides the standard example there is only *one* more extremal set, depicted in fig. 4 (left to the reader).

3. On $f(k,k;d)$ $(d \geq 1)$

Proposition 3.1.

$$\forall k, d \geq 1 \quad f(k,k;d) \geq \begin{cases} (d+1)k & d \text{ odd;} \\ dk+3 & k \geq 3, \; d \text{ even.} \end{cases} \tag{3.1}$$

Proof. for $d = 2m+1$ odd take $m+1$ skew lines in $\mathbb{R}^d$, $2k$ points on each line, to produce a set of cardinality $(m+1)2k = (d+1)k$ which is not of type $(k,k;d)$.

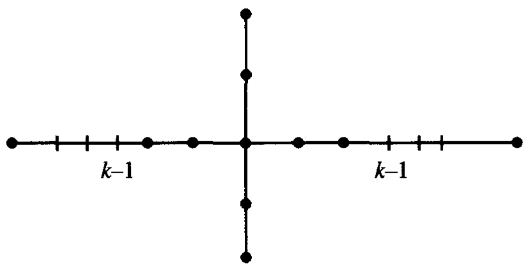

Fig. 5

For $d = 2m$ even take a planar set V of cardinality $2k+3$ which is not of type $(k,k;2)$, described in fig. 5, and add $(d-2)k = (m-1)\cdot 2k$ points on $m-1$ skew lines, which are affinely independent on aff(V), $2k$ points on each line. The resulting set has cardinality $dk+3$, and it is not of type $(k,k;d)$ (please check). ∎

The proof of the following result resembles that of Th. 2.1 in its use of Helly's Theorem. The method can be used to obtain upper bound for $f(k,k;d)$ such as in Proposition 3.3 below.

Proposition 3.2. *For $1 \leq k \leq 3$ and $d \geq 1$ $\quad f(k,k;d) = (d+1)k$.*

Remark. This is sharp in the sense that already for $k = 4$ and $d = 2$ it gives an uncorrect statement. However, by Proposition 3.1 (for d odd) and by Conjecture 2.3 it is conjectured that Proposition 3.2 holds for *all* $k \geq 1$ and odd d. (Case $d = 3$ of this conjecture was proved in Theorem 2.1.)

Proof. Inequality "$\geq$" follows from Proposition 3.1 above (please check).

Proof of "$\leq$" : For $d=1$ this is trivial. For $d=2$ this is the inequality $f(k,k;2) \leq 3k$, proved in [2] even for all $k \geq 1$. The sequel is by induction on d. Suppose that $d \geq 3$ and that the theorem is proved for all dimension $< d$. Let $V \subset \mathbb{R}^d$ be a spanning set, $\#V > (d+1)k$. We have to prove that if $1 \leq k \leq 3$, then V is of type (k,k). Meanwhile we do not assume $k \leq 3$ (only $k \geq 1$), since the discussion to follow until (3.5) (included) is valid for all $k \geq 1$. Denote by $\mathcal{H}$ the set of planes spanned by V which are *not* of type (k,k) (in particular, $\mathcal{H}$ contains all hyperplanes spanned by the facets of [V]).

For any hyperplane σ in $\mathcal{H}$ we can choose an open half-space, denoted by σ^-, whose boundary is σ, s.t. $\#(V \cap \sigma^-) < k$. Denote by σ^+ the complement of σ^-(σ^+ is a *closed* half-space which is *strong* in the sense introduced in the proof of Theorem 2.1). We note that the correspondence $\sigma \to \sigma^-$ is 1-1, although it may happen that there are two possible choices of σ^- for a given $\sigma \in \mathcal{H}$ (we choose only one half space). Let $\mathcal{F}$ be a family of closed half spaces σ^+ s.t. $\sigma \in \mathcal{H}$. Every member of $\mathcal{F}$ avoids at most $k-1$ points of V, hence the intersection of every $d+1$ members of $\mathcal{F}$ contains at least

$$\#V - (d+1)(k-1) \geq (d+1)k + 1 - (d+1)(k-1) = d+2 \tag{3.2}$$

points of V. If the intersection of every $d+1\sigma^+$'s has a full dimension, then by Helly's Theorem (full dimensional version) $\cap\mathcal{F}$ has a full dimension, and its interior is contained in int(conv V). In this case choose $z \in$ int $(\cap\mathcal{F})$ s.t. z is not contained in any hyperplane spanned by V, and let Δ be a cellular simplex (relatively to V; see 2.2 for definition) containing z in its interior. The intersection of the $d+1$ members of $\mathcal{F}$ which correspond to the $d+1$ facets of Δ is Δ, i.e., it contains only $d+1$ points of V, contradiction (to (3.2)).

Hence we may assume that there are $d+1$ members of $\mathcal{F}$ whose intersection has dimension $< d$. If the intersection was $d-1$ dimensional, then there would be two opposite half-spaces among the intersectors, which is impossible by the choice of the signs $+, -$. Hence the dimension of the intersection of any subfamily of $\mathcal{F}$ is either d or $\leq d-2$. Let $\mathcal{J} \subset \mathcal{F}$ be a minimal subfamily of $\mathcal{F}$ s.t. $\dim(\cap\mathcal{J}) < d$, i.e., $\dim(\cap\mathcal{J}) \leq d-2$ and every strict subfamily of $\mathcal{J}$ has a d-dimensional intersection. Put $j := \dim(\cap\mathcal{J})$. As mentioned above

$$(0 \leq) j \leq d-2. \tag{3.3}$$

It is easy to check that $\cap\mathcal{J}$ is a flat, and that $\#\mathcal{J} = d+1-j$. Hence

$$\begin{aligned} \#(V \cap (\cap\mathcal{J})) &\geq \#V - (d+1-j)(k-1) \\ &\geq (d+1)k + 1 - (d+1-j)(k-1) = d+1+1+j(k-1) \\ &= d+2-j+jk = (j+1)k + 1 + (d+1-j-k). \end{aligned} \tag{3.4}$$

By (3.3)

$$d+1-j-k \geq d+1-(d-2)-k = 3-k. \tag{3.5}$$

The foregoing discussion is valid for all $k \geq 1$ (it can be used also in the proof of Proposition 3.3 below). For $1 \leq k \leq 3$ we get by (3.4) and (3.5)

$$\#(V \cap (\cap\mathcal{J})) \geq (j+1)k + 1 \quad (1 \leq k \leq 3). \tag{3.6}$$

Put $J' = \text{aff}(V \cap (\cap\mathcal{J}))$, and $\dim J' = j' (\leq j)$. By (3.6) $\#(V \cap J') = \#(V \cap (\cap\mathcal{J})) \geq (j'+1)k+1$, and $\text{aff}(V \cap J') = J'$. By the induction hypothesis (for j') there is a $(j'-1)$-flat of type (k,k) in $V \cap J'$. This clearly implies that V is of type (k,k). ∎

As noted after (3.5), the foregoing proof can be used to obtain an upper bound for $f(k,k;d)$ which is of order $1\frac{1}{2}dk$, certainly not a sharp one, but seems to be not too far from the truth.

Proposition 3.3. *For all* $m \geq 1, k \geq 4$

(i) $$f(k,k;2m) \leq 3m(k-1)+2 \quad \textit{and}$$

(ii) $$f(k,k;2m+1) \leq (3m+1)(k-1)+4.$$

The proof resembles that of proposition 3.2, hence it is omitted.

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